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الجامعة الأردنية	تفاضل وتكامل ٢	الفصل الثاني
قسم الرياضيات	الاختبار الثاني	٢٠٠٨ – ٢٠٠٩
اسم الطالب:		مدرس المادة:
الرقم الجامعي		وقت المحاضرة:

[1] Find the area of the region bounded by the graphs:

$y = x + 6;\ \ y = x^3;\ \ 2y + x = 0.$ $\quad y = \frac{-x}{2}$

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$$A = \int_{-4}^{0} (x+6) - \left(\frac{-x}{2}\right) dx + \int_{0}^{2} x + 6 - x^3\, dx \qquad x^3 = x + 6 \Rightarrow x = 2$$

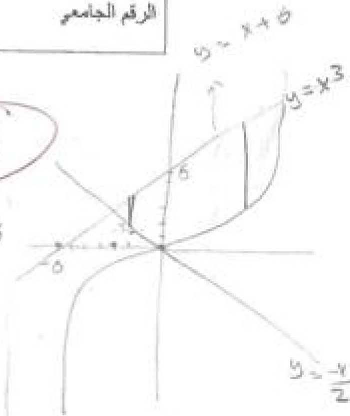
$$= \frac{x^2}{2} + 6x + \frac{1}{2}\frac{x^2}{2}\Big]_{-4}^{0} + \frac{x^2}{2} + 6x - \frac{x^4}{4}\Big]_{0}^{2}$$

$$= 0 - \left[\frac{16}{2} - 24 + \left(\frac{1}{2}\right)\left(\frac{16}{2}\right)\right] + \left[\frac{4}{2} + 12 - \frac{16}{4}\right] - 0$$

$$= -\left[8 - 24 + 4\right] + \left[2 + 12 - 4\right] = 12 + 10 = \boxed{22}$$

$-16 + 4$

-12



$\frac{-x}{2} = x + 6$

$-x = 2x + 12$

$-3x = 12 \Rightarrow \boxed{x = -4}$

[2] The region bounded by $y = x^2$ and $y = x + 2$ is revolved about the line $x = 3$. Find the volume of the solid generated.

$$V = \pi \int_{1}^{4} [f(y)]^2 - [g(y)]^2\, dy$$

$$V = \pi \int_{1}^{4} (3 - (y-2))^2 - (3 - \sqrt{y})^2\, dy$$

or

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$$V = 2\pi \int_{-1}^{2} (3 - x)\left[x + 2 - x^2\right] dx$$

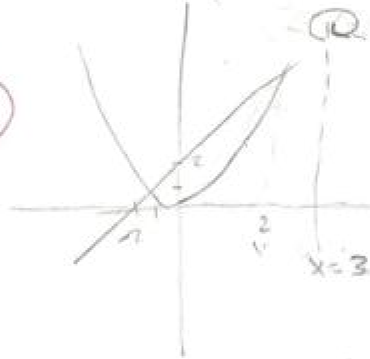
$$= 2\pi \int_{-1}^{2} (3-x)(x+2) - (3-x)x^2\, dx$$

$$= 2\pi \int_{-1}^{2} 3x + 6 - x^2 - 2x - 3x^2 + x^3\, dx$$

$$= 2\pi \left[\frac{3}{2}x^2 + 6x - \frac{x^3}{3} - x^2 - x^3 + \frac{x^4}{4}\right]_{-1}^{2}$$

$$2\pi\left[6 + 12 - \frac{8}{3} - 4 - 8 + 4\right] - \left[\frac{3}{2} - 6 + \frac{1}{3} - 1 + 1 + \frac{1}{4}\right]$$

$$2\pi\left[\frac{22}{3} + \frac{53}{12}\right] = (2\pi)\left(\frac{141}{12}\right) \qquad \frac{\pi\, 141}{6}$$



$x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$y = 4 \Leftarrow x = 2$

$y = 1 \Leftarrow x = -1$

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$F'(y) = \frac{4}{16}y^3 + \frac{1}{2}\frac{-2}{y^3}$

$\left[F'(y)\right]^2 = \frac{1}{16}y^6 + \frac{1}{y^6}$

[3] Find the arc length of the curve

$x = \frac{y^4}{16} + \frac{1}{2y^2}$ from $A(\frac{9}{8},-2)$ to $(\frac{9}{16},-1)$.

$\left[F'(y)\right]^2 = \left[\frac{1}{4}y^3 - \frac{1}{y^3}\right]^2$

$L = \int_{y_1}^{y_2} \sqrt{1+\left[F'(y)\right]^2}\,dx = L = \int_{-2}^{-1}\sqrt{1+\frac{1}{16}y^6}$

$= \frac{1}{16}y^6 - \frac{2}{4} + \frac{1}{y^6}$

$= \frac{1}{16}y^6 - \frac{1}{2} + \frac{1}{y^6}$

$= \int_{-2}^{-1}\sqrt{\left[\frac{1}{4}y^3 + \frac{1}{y^3}\right]^2}$

$1 + \left[F'(y)\right]^2 = \frac{1}{16}y^6 + \frac{1}{2} + \frac{1}{y^6}$

$= \int_{-2}^{-1} \frac{1}{4}y^3 + \frac{1}{y^3}\,dy$

$= \left[\frac{1}{4}y^3 + \frac{1}{y^3}\right]^2$

$= \left[\frac{1}{4}\left(\frac{y^4}{4}\right) + \left(\frac{1}{y^2}\right)\left(\frac{1}{-2}\right)\right]_{-2}^{-1}$

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$= \frac{y^4}{16} + \frac{1}{2y^2}\Big]_{-2}^{-1} = \left[\frac{1}{16} - \frac{1}{2}\right] - \left[\frac{16}{16} - \frac{1}{8}\right] = \frac{-7}{16} - \frac{7}{8} = \frac{-21}{16}$

[4] Find the **limit** of the following sequences:

(a) $a_n = \sqrt{n}\sin(\frac{\pi}{\sqrt{n}}) \Rightarrow \lim \frac{\sin\frac{\pi}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \sin\infty$

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$\lim_{n\to\infty}\sqrt{n} \cdot \sin\left(\lim \frac{\pi}{\sqrt{n}}\right)$

$\infty \cdot 0 = \text{zero}$

$\lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{\sqrt{n}}\right)}{\frac{1}{\sqrt{n}}} \times \frac{\pi}{\pi} = \pi \lim_{n\to\infty} \frac{\sin\frac{\pi}{\sqrt{n}}}{\frac{\pi}{\sqrt{n}}}$

(b) $a_n = \left(\frac{n}{1+n}\right)^{2n}$ $\Rightarrow \pi \lim_{z\to 0} \frac{\sin z}{z} = 1 \times \pi = \pi$

let $\frac{\pi}{\sqrt{n}} = z$

$\lim \left(\frac{n}{1+n}\right)^n \cdot \left(\frac{n}{1+n}\right)^n = e^{-1} \cdot e^{-1} = e^{-2}$

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let $n+1 = z$

$n = z - 1$

$\lim_{n\to\infty} \left(\frac{n+1-1}{1+n}\right)^n = \lim\left(\frac{n+1}{n+1} - \frac{1}{n+1}\right)^n = \lim\left(1 - \frac{1}{n+1}\right)^n$

$= \lim\left(1 - \frac{1}{z}\right)^{z-1} = \lim\left(1 - \frac{1}{z}\right)^z \cdot \lim_{n\to\infty}\left(1 - \frac{1}{z}\right)^{-1}$

$= e^1 \cdot 1 = e \cdot e = e^2$ $\quad e^{-1} \cdot e^{-1} = e^{-2}$

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[5] Find the sum $\sum_{n=1}^{\infty}\left(\frac{2}{e^n}+\frac{1}{(n+1)(n+2)}\right) \Rightarrow \sum_{n=1}^{\infty}\frac{2}{e^n} + \sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}$

① $\sum_{n-1=1}^{\infty} 2\left(\frac{1}{e}\right)^{n-1} = \sum_{n=0}^{\infty}(2)\left(\frac{1}{e}\right)\left(\frac{1}{e}\right)^{n} = \frac{2}{e}\left(\frac{1}{e}\right)^n$

$\left|\frac{1}{e}\right|<1$

$\Rightarrow sum = \frac{\frac{2}{e}}{1-\frac{1}{e}} = \frac{\frac{2}{e}}{\frac{e-1}{e}} = \frac{2}{e} \div \frac{e-1}{e} = \frac{2}{e} \times \frac{e}{e-1} = \boxed{\frac{2}{e-1}}$ $\boxed{\therefore \text{geometric}}$

② $\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)} = \frac{A}{n+1}+\frac{B}{n+2} = 1 \Rightarrow A(n+2)+B(n+1)=1$

$\Rightarrow n=-1 \Rightarrow \boxed{A=1}$

$\Rightarrow n=-2 \Rightarrow \boxed{B=-1}$

$\Rightarrow \sum_{n=1}^{\infty}\left[\frac{1}{n+1}-\frac{1}{n+2}\right] = \left[\left(\frac{1}{2}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}\right) \ldots\right.$

$\left.\ldots+\left(\cancel{\frac{1}{n}}-\cancel{\frac{1}{n+1}}\right)+\left(\cancel{\frac{1}{n+1}}-\frac{1}{n+2}\right)\right] = \left[\frac{1}{2}-\frac{1}{n+2}\right] \Rightarrow \lim_{n\to\infty}\left[\frac{1}{2}-\cancel{\frac{1}{n+2}}\right] = \frac{1}{2}$

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$\therefore sum = \boxed{\frac{2}{e-1}+\frac{1}{2}}$

[6] Test for convergence:

(a) $\sum_{n=1}^{\infty}\ln\left(\frac{n}{4n+1}\right)$

$\Rightarrow \lim \ln\left(\frac{n}{4n+1}\right) \Rightarrow \ln \lim_{n\to\infty}\left(\frac{n}{4n+1}\right) = \ln\frac{1}{4} \neq 0$ $\therefore$ dive

$\boxed{\text{By divergence test}}$

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(b) $\sum_{n=1}^{\infty}\frac{\cos(3n)}{1+n^4}$

Abs. con

$\Rightarrow \sum a_n$ $\Rightarrow$ By comparison test

$=\sum b_n$

$\frac{|\cos 3n|}{1+n^4} \leq \frac{1}{1+n^4}$

$\lim_{n\to\infty}\frac{1}{1+n^4}$

$\therefore \sum a_n$ conv

$\sum b_n$ conv

$\boxed{\cancel{\sum b_n \text{ Abs}}\ \cancel{\text{conv}}}$

let $c_n = \frac{1}{n^4} \Rightarrow \lim_{n\to\infty}\frac{b_n}{c_n} = \frac{1}{1+n^4} \times n^4 = \lim \frac{\cancel{n^4}}{\cancel{n^4}\left(\frac{1}{n^4}+1\right)} = 1$

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By limit comparison test

$\therefore c_n \Rightarrow$ conv $\Rightarrow \sum b_n \Rightarrow$ conv